

Gravitational waves during inflation from a 5D large-scale repulsive gravity model

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We investigate, in the transverse traceless (TT) gauge, the generation of the relic background of gravitational waves, generated during the early inflationary stage, on the framework of a large-scale repulsive gravity model. We calculate the spectrum of the tensor metric fluctuations of an effective 4D Schwarzschild-de-Sitter metric on cosmological scales. This metric is obtained after implementing a planar coordinate transformation on a 5D Ricci-flat metric solution, in the context of a non-compact Kaluza-Klein theory of gravity. We found that the spectrum is nearly scale invariant under certain conditions. One interesting aspect of this model is that it is possible to derive the dynamical field equations for the tensor metric fluctuations, valid not just at cosmological scales, but also at astrophysical scales, from the same theoretical model. The astrophysical and cosmological scales are determined by the gravity-antigravity radius, which is a natural length scale of the model, that indicates when gravity becomes repulsive in nature.

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I. INTRODUCTION

One of the greatest challenges of the modern cosmology is to explain the origin of the present day accelerated expansion. In spite of the many proposals, the problem is now still far to be closed. Cosmologists have followed basically two lines of research: the existence of a new exotic component in the content of the universe and gravitation theories alternative to general relativity. In this line of reasoning we have the well-known Λ -CDM model in which the cosmological constant is responsible for the acceleration in the expansion. Unfortunately, this model suffers from the cosmological constant problem and linked to it the cosmic coincidence problem is also a difficulty. Quintessence models have been proposed to explain or alleviate the cosmic coincidence problem: *why the value of the dark energy density today is comparable with the present matter energy density* [1–3]. The cosmic coincidence problem itself, motivated the appearance of interacting quintessence models [4–6]. The k-essence models are also an alternative [7–9]. None of them are free of difficulties and thus the accelerated expansion of the universe continue being an open problem for theoretical cosmology. In the second line of reasoning we can find IR modifications to general relativity, theories of gravity in more than four dimensions, and general modified theories of gravity [4]. However, some of them suffer from different kinds of instabilities [4].

The extra dimensional theories have been also a recourse scenario to address the present day accelerated expansion problem. Among others, we can mention for instance the brane-world theories [10–13], the M-theory [14, 15], and non-compact Kaluza-Klein (KK) theories [16–19]. The induced matter theory is regarded a non-compact KK theory in 5D, since the fifth extra dimension is assumed extended. This theory is mathematically supported by the embedding Campbell-Magaard theorem [20, 21]. The main idea in this theory is that matter in 4D can be geometrically induced from a 5D Ricci-flat metric. Thus, the theory considers a 5D geometrical vacuum defined by ${}^{(5)}R_{ab} = 0$, $a, b = 0, 1, 2, 3, 4$, which are the field equations of the theory. This idea has generated a new kind of cosmological

models on which accelerating expansion periods are included. On this setting, has been recently introduced a new cosmological model on which gravity manifests itself as attractive on astrophysical scales and as repulsive on cosmological scales [22]. In this model our the 4D universe is represented by an effective 4D Schwarzschild-de Sitter (SdS) metric. This metric is obtained after the implementation of a planar coordinate transformation on a 5D Ricci-flat SdS static metric, which is a 1-body solution to the Ricci-flat field equations of the theory. The scalar field fluctuations of the inflaton field in an early inflationary universe have been studied in the same physical framework in [23, 24]. In these works, it is obtained that the spectrum of the fluctuations at zeroth order is independent of the scalar field mass on Schwarzschild scales, while that on cosmological scales it exhibits a mass dependence [23, 24]. Our theory is basically an extension of the $\Lambda - CDM$ model and predicts an equation of state with $\omega = -1$. However it has the virtue that can describe the spectrum on both, cosmological and astrophysical scales. These results indicate that in this new model the fluctuations of the inflaton field during inflation are well described by repulsive gravity on cosmological scales. However, as it is well known, the inflationary mechanics in standard cosmology, also generates a background of gravitational waves [25]. Dark energy cosmological scenarios have been intensively studied in the last years [29]. The scenarios there described can explain the generation of gravitational waves on cosmological, but not on astrophysical scales. It is therefore useful and very instructive to study the relic background of gravitational waves generated during inflation on the framework of a large scale repulsive gravity model, since gravitons under such conditions must live imprints on the power spectrum and polarization of the cosmic microwave background radiation (CMBR). Thus, we may have possible ways to test this kind of large scale repulsive gravitational models.

In this letter we study the generation of relic background of cosmological gravitational waves during inflation, derived from a non-compact KK theory of gravity in the context of a repulsive gravity theory. We have organized the letter as follows. The Sect. II is devoted to obtain the 5D perturbed field equations. In Sect. III we study the propagation of the 5D tensor metric fluctuations along the extra coordinate, on cosmological scales. In Sect. IV, we study the 4D dynamics of gravitational waves on cosmological scales, induced from the 5D SdS spacetime and we calculate the 4D spectrum of gravitational waves. Finally in Sect. V, we give some final comments.

II. THE 5D PERTURBED FIELD EQUATIONS

Let us begin considering in the coordinate chart $\{T, R, \theta, \phi, \psi\}$, the 5D Ricci-flat metric [22]

$$dS_5^2 = \left(\frac{\psi}{\psi_0}\right)^2 \left[c^2 f(R) dT^2 - \frac{dR^2}{f(R)} - R^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] - d\psi^2, \quad (1)$$

where $f(R) = 1 - (2G\zeta\psi_0/Rc^2) - (R/\psi_0^2)$ is a dimensionless metric function, ψ is the spacelike and non-compact fifth extra coordinate¹. This metric is an extension to 5D spaces of the 4D SdS metric. T is a time-like coordinate, c is denoting the speed of light, R, θ, ϕ are the usual spherical polar coordinates, ψ_0 is an arbitrary constant with length units and the constant parameter ζ has units of $(mass)(length)^{-1}$. The metric (1) is static, however, it can be written on a dynamical coordinate chart $\{t, r, \theta, \phi, \psi\}$ by implementing the planar coordinate transformation [26]

$$R = ar \left[1 + \frac{G\zeta\psi_0}{2ar} \right]^2, \quad T = t + H \int^r dR \frac{R}{f(R)} \left(1 - \frac{2G\zeta\psi_0}{R} \right)^{-1/2}, \quad (2)$$

$a(t) = e^{Ht}$ being the scale factor and H the constant Hubble parameter. After doing so, the line element (1) may be expressed in terms of the conformal time τ in the form

$$dS_5^2 = \left(\frac{\psi}{\psi_0}\right)^2 [F(\tau, r) d\tau^2 - J(\tau, r) (dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2))] - d\psi^2, \quad (3)$$

where the metric functions $F(\tau, r)$ and $J(\tau, r)$ are given by

$$F(\tau, r) = a^2(\tau) \left[1 - \frac{G\zeta\psi_0}{2a(\tau)r} \right]^2 \left[1 + \frac{G\zeta\psi_0}{2a(\tau)r} \right]^{-2}, \quad J(\tau, r) = a^2(\tau) \left[1 + \frac{G\zeta\psi_0}{2a(\tau)r} \right]^4, \quad (4)$$

¹ In our notation conventions henceforth, latin indices $a, b =$ run from 0 to 4, whereas the rest of latin indices $i, j, n, l, \dots =$ run from 1 to 3.

with $d\tau = a^{-1}(\tau)dt$ and $a(\tau) = -1/(H\tau)$, so that the constant Hubble parameter satisfies

$$H = a^{-2} \frac{da}{d\tau}. \quad (5)$$

Notice that the metric in (3) is no more Ricci-flat. As it was shown in [22], for certain values of ζ and ψ_0 the metric in (1) has two natural horizons. The inner one is the analogous to the Schwarzschild horizon and the external one is the analogous to the Hubble Horizon. In the metric in (3) these horizons may of course be written in the new dynamical coordinate chart.

In order to study tensor metric fluctuations of the metric in (3), in the TT gauge, we use the 5D perturbed line element

$$dS_5^2|_{\text{pert}} = \left(\frac{\psi}{\psi_0}\right)^2 [F(\tau, x, y, z)d\tau^2 - J(\tau, x, y, z)(\delta_{ij} + \Pi_{ij})dx^i dx^j] - d\psi^2, \quad (6)$$

where we have written for simplicity the 3D spatially flat line element in cartesian coordinates, $\Pi_{ij}(\tau, x, y, z, \psi)$ is the transverse traceless tensor that describes the tensor metric fluctuations and satisfies: $\text{tr}(\Pi_{ij}) = \Pi^i_i = 0$ and $\partial_i \Pi^{ij} = 0$. Thus, we can write the spatial components of the metric as $g_{ij} = -(\psi/\psi_0)^2 J(\tau, x, y, z, \psi)(\delta_{ij} + \Pi_{ij})$, so that we can consider the linear approximation for the contravariant metric components $g^{ij} \simeq -(\psi/\psi_0)^{-2} J^{-1}(\tau, x, y, z, \psi)(\delta^{ij} - \Pi^{ij})$. Of course this perturbed metric is no Ricci-flat.

The dynamics for the tensor modes is obtained by using the linearized 5D Einstein equations in vacuum: $\delta R_{ab} = 0$, that for the case of the perturbed metric (6), become

$$\begin{aligned} & \frac{1}{4} \left(\frac{5J_{,i}F_{,j}}{J^2} - \frac{F_{,i}F_{,j}}{F^2} \right) \Pi^{ij} = 0 \\ & \frac{1}{2} \left(\frac{J}{F} \right) \ddot{\Pi}_{ij} - \frac{1}{4} \left(\frac{J\dot{F}}{F^2} - \frac{3\dot{J}}{F} \right) \dot{\Pi}_{ij} - \frac{J_{,n}}{J} \delta^{ln} \Pi_{ij,l} + \delta^{ln} \left(\frac{1}{4} \frac{F_{,l}}{F} + \frac{3}{2} \frac{J_{,l}}{J} \right) (\Pi_{ni,j} + \Pi_{nj,i} - \Pi_{ij,n}) - \frac{1}{4} \delta^{lm} \left(\frac{J_{,l}}{J} \delta_i^n + \right. \\ & \quad \left. + \frac{J_{,i}}{J} \delta_l^n - \frac{J_{,s}}{J} \delta^{ns} \delta_{il} \right) (\Pi_{mn,j} + \Pi_{mj,n} - \Pi_{nj,m}) - \frac{1}{4} \delta^{lm} \left(\frac{J_{,j}}{J} \delta_l^n + \frac{J_{,l}}{J} \delta_j^n - \frac{J_{,s}}{J} \delta^{ns} \delta_{lj} \right) (\Pi_{mi,n} + \Pi_{mn,i} - \Pi_{in,m}) + \\ & \quad + \frac{1}{2} \delta^{ln} (\Pi_{ni,jl} + \Pi_{nj,il} - \Pi_{ij,nl}) + \left[\frac{1}{2} \frac{\ddot{J}}{F} - \frac{1}{4} \frac{J\dot{F}}{F^2} - \frac{J}{\psi_0^2} - \frac{5}{4} \frac{J^2}{JF} - \left(\frac{J_{,nl}}{J} - \frac{J_{,n}J_{,l}}{J^2} \right) \delta^{ln} \right] \Pi_{ij} + \left(\frac{J_{,nl}}{J} - \frac{J_{,n}J_{,l}}{J^2} \right) \delta_{ij} \Pi^{nl} - \\ & \quad - \frac{1}{2} \left(\frac{F_{,l}J_{,n}}{FJ} + \frac{3J_{,l}J_{,n}}{J^2} \right) (\delta^{ln} \Pi_{ij} - \delta_{ij} \Pi^{ln}) - \frac{1}{2} \frac{J_{,m}}{J} \left(\frac{J_{,j}}{J} \delta_i^n + \frac{J_{,i}}{J} \delta_j^n - \frac{J_{,s}}{J} \delta^{ns} \delta_{ij} \right) (\delta^{lm} \Pi_{ln} - \delta_{ln} \Pi^{lm}) + \\ & \quad + \frac{1}{2} \frac{J_{,m}}{J} \left(\frac{J_{,l}}{J} \delta_i^n + \frac{J_{,i}}{J} \delta_l^n - \frac{J_{,s}}{J} \delta^{ns} \delta_{il} \right) (\delta^{lm} \Pi_{nj} - \delta_{nj} \Pi^{lm}) + \frac{1}{2} \frac{J_{,m}}{J} \left(\frac{J_{,j}}{J} \delta_l^n + \frac{J_{,l}}{J} \delta_j^n - \frac{J_{,s}}{J} \delta^{ns} \delta_{lj} \right) (\delta^{lm} \Pi_{in} - \delta_{in} \Pi^{lm}) - \\ & \quad - \frac{4\psi}{\psi_0^2} J \Pi_{ij,\psi} - \frac{1}{2} \left(\frac{\psi}{\psi_0} \right)^2 J \Pi_{ij,\psi\psi} = 0, \end{aligned} \quad (7)$$

where the dot is denoting derivative with respect to the conformal time τ . The main difference with respect to the tensor metric fluctuation analysis usually implemented in general relativity, relies in the fact that the equations (7) and (8) describe the dynamics of the tensor perturbations $\Pi_{ij}(\tau, x, y, z, \psi)$ on both astrophysical and cosmological scales. This feature is proper of the 5D model we are working on. By the time, we will focus only on the dynamics at cosmological scales.

III. PROPAGATION OF THE 5D TENSOR FLUCTUATIONS ALONG THE EXTRA COORDINATE ON COSMOLOGICAL SCALES

On cosmological scales the next condition is satisfied

$$\frac{G\zeta\psi_0}{2a(\tau)r_H} \ll 1, \quad (9)$$

where r_H denotes the value of the radial coordinate at the horizon entry. The perturbed field equations (8) with the condition (7) reduce in this limit case to

$$\frac{1}{2} \left(\frac{J}{F} \right) \ddot{\Pi}_{ij} - \frac{1}{4} \left(\frac{J\dot{F}}{F^2} - \frac{3\dot{J}}{F} \right) \dot{\Pi}_{ij} - \frac{1}{2} \delta^{nl} \Pi_{ij, nl} + \left(\frac{1}{2} \frac{\ddot{J}}{F} - \frac{1}{4} \frac{\dot{J}\dot{F}}{F^2} - \frac{J}{\psi_0^2} - \frac{5}{4} \frac{\dot{J}^2}{JF} \right) \Pi_{ij} - 4 \frac{\psi}{\psi_0^2} J \Pi_{ij, \psi} - \frac{1}{2} \left(\frac{\psi}{\psi_0^2} \right) J \Pi_{ij, \psi\psi} = 0. \quad (10)$$

Notice that the left hand side of the equation (7) becomes identically zero on cosmological scales. Using the fact that on this length scales $J(\tau, r) \simeq a(\tau)^2$ and $F(\tau, r) \simeq a(\tau)^2$, the expression (10) in spherical polar coordinates (r, θ, φ) reads

$$\ddot{\Pi}_{ij} + 2\mathcal{H}\dot{\Pi}_{ij} - \nabla_{\bar{r}}^2 \Pi_{ij} + 2 \left(\dot{\mathcal{H}} - 4\mathcal{H}^2 - \frac{a^2}{\psi_0^2} \right) \Pi_{ij} - a^2 \left[\frac{8\psi}{\psi_0^2} \star \Pi_{ij} + \left(\frac{\psi}{\psi_0} \right)^2 \star\star \Pi_{ij} \right] = 0, \quad (11)$$

being $\mathcal{H}(\tau) = \dot{a}(\tau)/a(\tau)$ the conformal Hubble parameter, $\nabla_{\bar{r}}^2$ denoting the Laplacian operator in spherical polar coordinates and where the star (\star) denotes derivative with respect to ψ . Now, in order to implement the canonical quantization of (11), let us introduce the Fourier expansion

$$\Pi_j^i(\tau, \bar{r}, \psi) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_r \int dk_\psi \sum_{\alpha=+, \times}^{(\alpha)} e_j^i \left[a_{k_r k_\psi}^{(\alpha)} e^{i\bar{k}_r \cdot \bar{r}} \xi_{k_r k_\psi}(\tau, \psi) + a_{k_r k_\psi}^{(\alpha)\dagger} e^{-i\bar{k}_r \cdot \bar{r}} \xi_{k_r k_\psi}^*(\tau, \psi) \right], \quad (12)$$

with the asterisk ($*$) denoting complex conjugate, α counting the number of degrees of freedom, and where $a_{k_r k_\psi}^{(\alpha)\dagger}$ and $a_{k_r k_\psi}^{(\alpha)}$ are the creation and annihilation operators, respectively, that satisfy the commutator algebra

$$\left[a_{k_r k_\psi}^{(\alpha)}, a_{k'_r k'_\psi}^{(\alpha')} \right] = g^{\alpha\alpha'} \delta^{(3)}(\bar{k}_r - \bar{k}'_r) \delta(k_\psi - k'_\psi), \quad (13)$$

$$\left[a_{k_r k_\psi}^{(\alpha)}, a_{k'_r k'_\psi}^{(\alpha')\dagger} \right] = \left[a_{k_r k_\psi}^{(\alpha)\dagger}, a_{k'_r k'_\psi}^{(\alpha')\dagger} \right] = 0, \quad (14)$$

and where $^{(\alpha)}e_{ij}$ is the polarization tensor that obeys

$$^{(\alpha)}e_{ij} = ^{(\alpha)}e_{ji}, \quad k^i ^{(\alpha)}e_{ij} = 0, \quad (15)$$

$$^{(\alpha)}e_{ii} = 0, \quad ^{(\alpha)}e_{ij}(-\bar{k}_r) = ^{(\alpha)}e_{ij}^*(\bar{k}_r). \quad (16)$$

In this manner, substituting (12) in (11) we obtain

$$\ddot{\xi}_{k_r k_\psi} + 2\mathcal{H}\dot{\xi}_{k_r k_\psi} + \left[k_r^2 + 2(\dot{\mathcal{H}} - 4\mathcal{H}^2 - \psi_0^{-2} a^2) \right] \xi_{k_r k_\psi} - a^2 \left[\frac{8\psi}{\psi_0^2} \star \xi_{k_r k_\psi} + \left(\frac{\psi}{\psi_0} \right)^2 \star\star \xi_{k_r k_\psi} \right] = 0, \quad (17)$$

which is the equation that describes the dynamics of the 5D tensor modes $\xi_{k_r k_\psi}(\tau, \psi)$.

We decompose the 5D tensor modes $\xi_{k_r k_\psi}(\tau, \psi)$ into the KK-modes

$$\xi_{k_r k_\psi}(\tau, \psi) = \int d\tilde{m} \xi_{\tilde{m}}(\tau) \Omega_{\tilde{m}}(\psi), \quad (18)$$

and the equation (17) reduces to the system

$$\ddot{\xi}_{\tilde{m}} + 2\mathcal{H}\dot{\xi}_{\tilde{m}} + \left[k_r^2 + 2(\dot{\mathcal{H}} - 4\mathcal{H}^2 - \psi_0^{-2} a^2) + \tilde{m}^2 \right] \xi_{\tilde{m}} = 0, \quad (19)$$

$$\left(\frac{\psi}{\psi_0} \right)^2 \star\star \Omega_{\tilde{m}} + \frac{8\psi}{\psi_0^2} \star \Omega_{\tilde{m}} + \tilde{m}^2 \Omega_{\tilde{m}} = 0, \quad (20)$$

where the parameter \tilde{m}^2 corresponds to the square of the KK-mass measured by a 5D observer. To simplify the structure of (20) we introduce the field transformation: $\Omega_{\tilde{m}}(\psi) = (\psi_0/\psi)^4 P_{\tilde{m}}(\psi)$, and the equation (20) results to be

$$\psi^2 \star\star P_{\tilde{m}} - (12 - \tilde{m}^2 \psi_0^2) P_{\tilde{m}} = 0. \quad (21)$$

After resolve this equation, we obtain

$$P_{\tilde{m}}(\psi) = C_1 \psi^{\gamma_1} + C_2 \psi^{\gamma_2}, \quad (22)$$

where $\gamma_1 = (1/2)[1 + \sqrt{49 - 4\tilde{m}^2\psi_0^2}]$, $\gamma_2 = (1/2)[1 - \sqrt{49 - 4\tilde{m}^2\psi_0^2}]$ and C_1, C_2 are integration constants.

In order to study the stability of the KK-modes, let us rewrite the equation (20) in terms of the conformal spatial fifth coordinate: $u = \ln(\psi/\psi_0)$. Using the auxiliary field transformation: $\Omega_{\tilde{m}}(u) = e^{-(7/2)u} L_{\tilde{m}}(u)$, we obtain

$$\frac{d^2 L_{\tilde{m}}}{du^2} + \left(\tilde{m}^2 \psi_0^2 - \frac{49}{4} \right) L_{\tilde{m}} = 0. \quad (23)$$

The stability of the KK-modes depends directly on the stability of the solutions of the equations (19) and (20). One can see from (23) that for $\tilde{m}^2 \psi_0^2 > 49/4$, the redefined modes L_m are coherent (stable) on the ultraviolet (UV) sector. When $\tilde{m}^2 \psi_0^2 < 49/4$ those L_m -modes are unstable and diverge when u tends to infinity. The L_m -mode with $\tilde{m} = 0$ is stable. However, notice that even when the L_m -modes are unstable for $\tilde{m}^2 \psi_0^2 < 49/4$, the modes $\Omega_{\tilde{m}}(u) = e^{-(7/2)u} L_{\tilde{m}}(u)$, for any \tilde{m} real, never diverges. On the other hand, in the case of the equation (19), using the transformation $\xi_{\tilde{m}} = \Theta_{\tilde{m}} \exp(-\int \mathcal{H} d\tau)$, we obtain

$$\ddot{\Theta}_{\tilde{m}} + [k_r^2 + \dot{\mathcal{H}} - 9\mathcal{H}^2 - 2\psi_0^{-2}a^2 + \tilde{m}^2] \Theta_{\tilde{m}} = 0. \quad (24)$$

It follows from this equation that the modes $\Theta_{\tilde{m}}$ are stable for

$$k_r^2 > 9\mathcal{H}^2 + 2\psi_0^{-2}a^2 - \dot{\mathcal{H}} - \tilde{m}^2 \geq 0. \quad (25)$$

This condition is satisfied for $\tilde{m} \ll 1$. However, for modes with a bigger mass \tilde{m} and $k_r^2 \ll 1$ the condition (25) is no valid and hence, on very large scales, the modes $\Theta_{\tilde{m}}$ are unstable.

IV. THE 4D INDUCED DYNAMICS OF GRAVITATIONAL WAVES ON COSMOLOGICAL SCALES

In order to derive the 4D dynamics of the inflationary universe on cosmological scales induced by the 5D field equations (11), let us assume that the 5D spacetime can be foliated by a family of hypersurfaces $\Sigma : \psi = \psi_0$, where our usual 4D universe is represented by a leaf member of the foliation Σ . On every leaf the metric induced by (3) take the form

$$dS_4^2 = F(\tau, r) dr^2 - J(\tau, r) [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)], \quad (26)$$

where the metric functions $F(\tau, r)$ and $J(\tau, r)$ are now given in terms of the physical mass $m = \zeta \psi_0$ (introduced by the first time in [22]) as it is shown in the expressions

$$F(\tau, r) = a^2(\tau) \left[1 - \frac{Gm}{2a(\tau)r} \right]^2 \left[1 + \frac{Gm}{2a(\tau)r} \right]^{-2}, \quad J(\tau, r) = a^2(\tau) \left[1 + \frac{Gm}{2a(\tau)r} \right]^4, \quad (27)$$

which are valid for $r > Gm/(2a)$. The induced metric (26) describes a black hole immersed in an expanding universe, where the expansion is driven by a kind of cosmological constant, whose value depends of the ψ_0 -value, which is related to the Hubble constant through: $\psi_0 = c^2/H$, with c denoting the speed of light [22]. The condition that determines cosmological scales (9), can be now expressed in terms of this physical mass m , in the form

$$\frac{Gm}{2a(\tau)r_H} \ll 1, \quad (28)$$

so that in this limit the functions: $F(\tau, r)|_{r \gg Gm/(2a)} \simeq F(\tau) = a^2(\tau)$, $J(\tau, r)|_{r \gg Gm/(2a)} \simeq J(\tau) = a^2(\tau)$, become independent of r , and describe an isotropic and asymptotically homogenous expanding universe with a scale factor $a(\tau)$, in agreement with that one expects in a FRW cosmology. In particular, when $a(\tau) = -1/(H\tau)$, we are dealing with a de Sitter (inflationary expansion) with an equation of state $p/\rho = \omega = -1$. This topic was studied with detail in [24]. The realization of a reheating epoch after inflation from this asymptotic de Sitter metric, with particles and gravitons creation included, was studied in [30].

As it was shown in [22], a very interesting aspect of (26), is the existence of a length scale that separates regions where gravity is attractive (small astrophysical scales), from regions where it behaves as repulsive (cosmological scales). This length scale is known as the gravitational-antigravitational radius, which in the coordinate chart (T, R) is determined by the expression

$$R_{ga} = (Gm\psi_0^2)^{1/3}. \quad (29)$$

In the dynamical coordinate chart (τ, r) , we have a new gravitational-antigravitational radius r_{ga} given by

$$r_{ga} = \frac{1}{2a(\tau)} \left[R_{ga} - Gm + \sqrt{R_{ga}^2 - 2GmR_{ga}} \right]. \quad (30)$$

It can be easily seen from this expression that r_{ga} is positive when $R_{ga} \geq 2Gm$, and due to (29) such a condition can be written as

$$\epsilon = \frac{mH}{M_p^2} \leq \frac{1}{2\sqrt{2}} \simeq 0.353553, \quad (31)$$

where we have used $G = M_p^{-2}$. It follows from this last condition that the value of the fifth coordinate ψ_0 is restricted by: $\psi_0 \geq (2\sqrt{2}m)/M_p^2$.

In order to derive the 4D field equations induced on Σ by the 5D geometry, now we consider a separable 5D field

$$\Pi_{ij}(\tau, \bar{r}, \psi) = h_{ij}(\tau, \bar{r})\Gamma(\psi). \quad (32)$$

Thus, the part of (11) that depends only on the extra coordinate ψ , has the form

$$\left(\frac{\psi}{\psi_0} \right)^2 \Gamma^{**} + \frac{8\psi}{\psi_0^2} \Gamma^* - \lambda_0 \Gamma = 0, \quad (33)$$

λ_0 being a separation constant. The general solution of this equation reads

$$\Gamma(\psi) = B_1 \psi^{\alpha_1} + B_2 \psi^{\alpha_2}, \quad (34)$$

where B_1 and B_2 are integration constants and, $\alpha_1 = (1/2)(-7 + \sqrt{49 + 4\lambda_0\psi_0^2})$, $\alpha_2 = (1/2)(-7 - \sqrt{49 + 4\lambda_0\psi_0^2})$. With the help of (32), it follows from (11) that the 4D induced field equations on $\Sigma : \psi = \psi_0$ have the form

$$\ddot{h}_{ij} + 2\mathcal{H}\dot{h}_{ij} - \nabla_{\bar{r}}^2 h_{ij} + \left[2(\dot{\mathcal{H}} - 4\mathcal{H}^2) - \frac{\sigma}{\psi_0^2} a^2 \right] h_{ij} = 0, \quad (35)$$

where $\sigma = (2\psi_0^2 + \gamma_0)\psi_0^2$ with $\gamma_0 = [(8/\psi_0)(\dot{\Omega}/\Omega) + (\dot{\Omega}^*/\Omega^*)]|_{\psi=\psi_0}$, which for $B_2 = 0$ in (34), reduces to $\sigma = 2 + \alpha_1(\alpha_1 - 7)$. It is important to notice that the last term on the left hand side of (35) is a new correction coming from the 5D geometry. If we introduce the auxiliary field w_{ij} defined by $h_{ij}(\tau, r) = a^{-1}(\tau)w_{ij}(\tau, r)$, the equation (35), yields

$$\ddot{w}_j^i - \nabla^2 w_j^i + \left(\dot{\mathcal{H}} - 9\mathcal{H}^2 - \frac{\sigma}{\psi_0^2} a^2 \right) w_j^i = 0. \quad (36)$$

To quantize the field equation (36), we use the 5D Fourier expansion (12) specialized on Σ

$$w_j^i(\tau, \bar{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_r dk_\psi \sum_{\alpha=+, \times} (\alpha) e_j^i \left[a_{k_r k_\psi}^{(\alpha)} e^{i\bar{k}_r \cdot \bar{r}} \xi_{k_r k_\psi}(\tau, \psi) + a_{k_r k_\psi}^{(\alpha)\dagger} e^{-i\bar{k}_r \cdot \bar{r}} \xi_{k_r k_\psi}^*(\tau, \psi) \right] \delta(k_\psi - k_{\psi_0}), \quad (37)$$

whose integration on k_ψ yields the 4D expansion

$$w_j^i(\tau, \bar{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_r \sum_{\alpha=+, \times} (\alpha) e_j^i \left[a_{k_r}^{(\alpha)} e^{i\bar{k}_r \cdot \bar{r}} \xi_{k_r}(\tau) + a_{k_r}^{(\alpha)\dagger} e^{-i\bar{k}_r \cdot \bar{r}} \xi_{k_r}^*(\tau) \right], \quad (38)$$

where we have done the formal identifications: $\xi_{k_r}(\tau) = \xi_{k_r k_{\psi_0}}(\tau)$, $a_{k_r}^{(\alpha)} = a_{k_r k_{\psi_0}}^{(\alpha)}$, $a_{k_r}^{(\alpha)\dagger} = a_{k_r k_{\psi_0}}^{(\alpha)\dagger}$, with the creation and annihilation operators $a_{k_r}^{(\alpha)\dagger}$ and $a_{k_r}^{(\alpha)}$, respectively. These operators satisfy the algebra

$$\left[a_{k_r}^{(\alpha)}, a_{k_r'}^{(\alpha')\dagger} \right] = g^{\alpha\alpha'} \delta^{(3)}(\bar{k}_r - \bar{k}_r'), \quad (39)$$

$$\left[a_{k_r}^{(\alpha)}, a_{k_r'}^{(\alpha')} \right] = \left[a_{k_r}^{(\alpha)\dagger}, a_{k_r'}^{(\alpha')\dagger} \right] = 0, \quad (40)$$

and where the same properties for the polarization tensor ${}_{(\alpha)}e^{ij}$, given by (15) and (16), hold.

The quantization procedure requires that the modes ξ_{k_r} must satisfy the commutation relation

$$\left[\xi_{k_r}(\tau, \bar{r}), \dot{\xi}_{k_r'}^*(\tau, \bar{r}') \right] = i\delta^{(3)}(\bar{r} - \bar{r}'). \quad (41)$$

Therefore, with the use of (38), the expression (41) complies with the normalization condition

$$\xi_{k_r} \dot{\xi}_{k_r}^* - \xi_{k_r}^* \dot{\xi}_{k_r} = i, \quad (42)$$

on the UV-sector. In this manner, if the modes ξ_{k_r} satisfy (42), automatically (41) will hold. After inserting (38) in (36), we obtain

$$\ddot{\xi}_{k_r} + \left[k_r^2 - \left(9\mathcal{H}^2 - \dot{\mathcal{H}} + \frac{\sigma}{\psi_0^2} a^2 \right) \right] \xi_{k_r} = 0. \quad (43)$$

The general solution for this equation is given by

$$\xi_{k_r}(\tau) = A_1 \sqrt{-\tau} \mathcal{H}_{\nu_T}^{(1)}(-k_r \tau) + A_2 \sqrt{-\tau} \mathcal{H}_{\nu_T}^{(2)}(-k_r \tau), \quad (44)$$

$\mathcal{H}_{\nu_T}^{(1,2)}$ being the first and second kind Hankel functions with order $\nu_T = [1/(2\psi_0 H)] \sqrt{33\psi_0^2 H^2 + 4\sigma}$. After make the Bunch-Davies normalization [27], we obtain the normalized solution that satisfies (42) in the UV-sector. It is

$$\xi_{k_r}(\tau) = i \frac{\sqrt{\pi}}{2} \sqrt{-\tau} \mathcal{H}_{\nu_T}^{(2)}[-k_r \tau]. \quad (45)$$

Once we have calculated the analytic expression for the quantum tensor modes $\xi_{k_r}(\tau)$, we are now in position to obtain the spectrum of gravitational waves induced on our 4D spacetime Σ .

A. The 4D spectrum of gravitational waves

In order to obtain the 4D spectrum of gravitational waves on Super-Hubble (cosmological) scales, let us calculate the amplitude of the 4D tensor metric fluctuations on the IR-sector ($-k_r \tau \ll 1$), which is given by

$$\langle h^2(\tau) \rangle_{IR} = \frac{4a^{-2}(\tau)}{\pi^2} \int_0^{\epsilon k_H} \frac{dk_r}{k_r} k_r^3 [\xi_{k_r}(\tau) \xi_{k_r}^*(\tau)]|_{IR}, \quad (46)$$

where $\epsilon = k_{max}^{IR}/k_p \ll 1$ is a dimensionless parameter, being $k_{max}^{IR} = k_H(\tau_i) = \sqrt{9\mathcal{H}^2 - \dot{\mathcal{H}} + (\sigma/\psi_0^2)a^2} = \sqrt{10 + [\sigma/(\psi_0^2 H^2)]} \tau^{-1}$ the wave number related to the Hubble radius at the conformal time τ_i , which corresponds to the time when the modes re-enter to the horizon, and k_p is the Planckian wave number. During inflation $H = 0.5 \cdot 10^{-9} M_p$ and thus the values of ϵ range from 10^{-5} to 10^{-8} , corresponding to the number of e -foldings: $N_e = 63$ [28]. Considering the asymptotic expansion for the Hankel function $\mathcal{H}_{\nu_T}^{(2)}[x] \simeq -(i/\pi) \Gamma(\nu_T) (x/2)^{-\nu_T}$, the expressions (45) and (46) yield

$$\langle h^2(\tau) \rangle_{IR} = \frac{2^{2\nu_T-1}}{\pi^3} H^2 \Gamma^2(\nu_T) (-\tau)^{3-2\nu_T} \int_0^{\epsilon k_H} \frac{dk_r}{k_r} k_r^{3-2\nu_T}. \quad (47)$$

Performing the integration in (47) we arrive to

$$\langle h^2(\tau) \rangle_{IR} = \frac{2^{2\nu_T-1}}{\pi^3} \frac{\Gamma^2(\nu_T)}{3-2\nu_T} H^2 \epsilon^{3-2\nu_T} (-k_r \tau)^{3-2\nu_T}, \quad (48)$$

which in terms of the cosmic time t can be written in the form

$$\langle h^2(t) \rangle_{IR} = \frac{2^{1+2\nu_T}}{\pi} \frac{\Gamma^2(\nu_T)}{3-2\nu_T} \left(\frac{H}{2\pi} \right)^2 \epsilon^{3-2\nu_T} \left(\frac{k_H}{aH} \right)^{3-2\nu_T}. \quad (49)$$

The power spectrum for gravitational waves can be then extracted from (49), resulting

$$\mathcal{P}_g(k_H) = \frac{2^{1+2\nu_T}}{\pi} \Gamma^2(\nu_T) \left(\frac{H}{2\pi} \right)^2 \left(\frac{k_r}{aH} \right)^{3-2\nu_T} \Big|_{k_H}. \quad (50)$$

This spectrum is nearly scale invariant ($\nu_T \simeq 3/2$) for $\sigma \simeq -6\psi_0^2 H^2$. This value of σ corresponds to $\alpha_1 \simeq \frac{7}{2} \pm \sqrt{41 - 24\psi_0^2 H^2}$. Notice that the last term in the brackets of (35): $-\frac{4\sigma}{\psi_0^2 H^2}$, is the responsible to broke the scale invariance in the spectrum of gravitational waves. In our formalism this term appears due to the existence of the extra non-compact space-like dimension. However, in the standard 4D inflationary theory, a possible origin of this term could be in the coupling of the inflaton to new sectors of light degrees of freedom which are closely spaced along its trajectory[31]. From WMAP5[32] data, we obtain that the acceptable values of $-\frac{\sigma}{\psi_0^2 H^2}$ is in the range

$$5.89649 > -\frac{\sigma}{\psi_0^2 H^2} > 5.7659, \quad (51)$$

once we require that the spectral index to be in the range: $n_s = 0.963_{-0.015}^{+0.014}$, for a tensor index $n_T = 1 - n_s$. In the same context, it can be shown that the spectral index for scalar metric fluctuations is given by: $n_s \simeq 4 - 2\nu_s \simeq 4 - \sqrt{9 + 16k_{\psi_0}^2 H^{-2}}$. If we define, in analogy with the definition of slow-roll parameters of 4D inflationary models, the parameter $\theta = 16k_{\psi_0}^2/H^2 \ll 1$, the relation $2 - \nu_s \simeq \theta$ holds. Hence, the scalar-tensor ratio $r_{ST} = A_T^2(k_r)/A_S^2(k_r)$ satisfies: $r_{ST} \simeq (1/8)\theta \simeq [k_{\psi_0}^2/(2H^2)]$. Thus, if the parameter k_{ψ_0} varies in the interval $0 < k_{\psi_0}^2 \leq (0.15)^2 H^2$, then the scalar-tensor ratio ranges in the interval $0 < r_{ST} \leq 0.0112$ [30, 33, 34]. With PLANCK experiment, through its B-mode polarization of relic gravitational waves observations, will be able to reject models with $r_{ST} > 0.095$ [35]. However, the restrictions on $k_{\psi_0}^2$ must be more carefully established in order to have a more accurate prediction for the scalar-tensor ratio r_{ST} in this large-scale repulsive-gravity model, but it goes beyond the scope of this paper.

V. FINAL COMMENTS

In this letter we have studied the relic background of gravitational waves generated during inflation, on the framework of a cosmological model in which gravity manifests as attractive on astrophysical scales and as repulsive on cosmological scales. We have obtained particularly the spectrum of gravitational waves on cosmological scales.

We use a static and Ricci-flat 5D metric to derive, via a planar coordinate transformation, a dynamical 5D metric that can be considered as a 5D extension of a Schwarzschild-de-Sitter metric. The fifth extra dimension is considered as non-compact and our universe is represented by a 4D spacetime locally and isometrically embedded into a 5D ambient space metrically described by the field equations: ${}^{(5)}R_{ab} = 0$. The cosmological and astrophysical scales are, in this model, defined in terms of a length scale called the gravity-antigravity radius r_{ga} [22]. Thus, the astrophysical scale, normally used in astrophysical models, coincides with the region defined by $r < r_{ga}$ and gravity is attractive here, whereas the cosmological scale coincides with the region $r > r_{ga}$ in which gravity is repulsive in nature [22–24]. Dynamical 5D field equations for the tensor metric fluctuations in the TT-gauge of the non-static metric (3) are obtained (see the equations (7) and (8)). The interesting of these equations is that they are valid for both astrophysical and cosmological scales. In 4D the induced effective field equations contain an additional term which depends of a constant parameter σ (see the expression (35) and the definitions below it). This additional term is a new contribution coming from the 5D geometry. The spectrum of gravitational waves, given by the equation (50), resulted nearly scale invariant when the parameter σ in the extra term of (35) satisfies $\sigma \simeq -6\psi_0^2 H^2$. The amplitude of the 4D square tensor metric fluctuations on the IR-sector, given by (49), behaves in a similar manner as they do in standard 4D inflationary models. We consider that these results are of great utility to test the viability of this new class of large-scale repulsive gravity models.

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